On Colorings Induced by Low-Index Subgroups of Some Hyperbolic Triangle Group

Ruween Nicky C. Guzman[1*], Lawrence A. Eclarin[1]

[1] Mariano Marcos State University, Philippines
[*] Corresponding author's email: rcguzman@mmsu.edu.ph

ABSTRACT

Tilings have been studied in mathematics since ancient times, traditionally within the framework of Euclidean geometry. In recent years, growing interest has emerged in non-Euclidean tilings, particularly those in spherical and hyperbolic geometry, where regular convex polygons generate highly structured and visually striking patterns. Beyond their theoretical importance, hyperbolic tilings offer meaningful applications in areas such as artistic design, architectural pattern formation, mathematical education, and computer-based visualization, where symmetry and color play a central role. This study investigates colorings induced by low-index subgroups of the hyperbolic triangle group *732. Specifically, it constructs right coset colorings of the group, identifies the structural components of its low-index subgroups, and establishes subgroup properties in relation to their induced colorings. Computational tools were employed using GAP (Groups, Algorithms, and Programming) to generate low-index subgroups of the triangle group *732 and to produce the corresponding right coset colorings of the hyperbolic plane. The results show that tiling the entire hyperbolic plane can be achieved by appropriately joining the vertices of a tiling's fundamental region to form a single, complete rotation. These findings enhance the understanding of symmetry, subgroup structure, and color organization in hyperbolic geometry. Moreover, the resulting colored tilings provide a mathematical foundation for creating complex visual patterns that may be adapted for architectural surfaces, decorative designs, and interactive educational materials. The study demonstrates how abstract group-theoretic concepts can be translated into visually meaningful representations, thereby bridging pure mathematics with applied and creative disciplines. Future research is recommended to explore induced colorings of low-index subgroups of other hyperbolic triangle groups and related hyperbolic structures.

***Keywords**: Coloring, Hyperbolic Triangle, Low-Index Subgroup*

Introduction

Across cultures and historical periods, mathematics and art have been deeply connected through the pursuit of pattern, symmetry, and structure. From ornamental designs and mosaics to architectural decorations, mathematical principles have long guided aesthetic expression. The formal mathematical study of color symmetry emerged in the early 20th century and was significantly advanced by Shubnikov's theory of antisymmetry, later extended by Belov and collaborators to multicolor symmetry systems. These developments, together with the visual impact of Escher's tessellations, established color symmetry as an important area of mathematical inquiry (Amidror, 2009; Washburn & Crowe, 2004). Much of the existing literature on color symmetry and tilings has focused on Euclidean geometry, where the classification of symmetry groups and colorings is well understood. However, non-Euclidean geometries, particularly hyperbolic geometry, possess fundamentally richer symmetry structures due to their negative curvature. Hyperbolic tilings admit infinitely many symmetry types and allow regular polygons to meet in configurations impossible in the Euclidean plane (Coxeter, 1957). Despite this potential, systematic studies of induced colorings arising from subgroup structures in hyperbolic triangle groups remain limited, especially those that explicitly connect algebraic constructions with geometric and visual outcomes. This gap restricts both theoretical progress and the translation of hyperbolic symmetry into applied domains. Among hyperbolic symmetry groups, the hyperbolic triangle group 732 plays a central role. It is generated by reflections in the sides of a triangle with angles $\pi/7$, $\pi/3$, and $\pi/2$, making it one of the simplest yet most symmetric hyperbolic triangle groups. Its low-index subgroups yield a wide variety of nontrivial coset partitions, making *732 particularly suitable for studying induced colorings that reflect both algebraic structure and geometric regularity (Magnus, Karrass, & Solitar, 1976; Coxeter & Moser, 1980). Despite its importance, detailed investigations of colorings induced by low-index subgroups of 732 are scarce, highlighting the need for focused research in this area. This study addresses this gap by investigating colorings induced by low-index subgroups of the hyperbolic triangle group 732. Specifically, it aims to obtain right coset colorings of *732, identify the structural components of its low-index subgroups, and establish subgroup properties in relation to their induced colorings using computational methods. Hyperbolic colorings provide new frameworks for textile and surface design, where repeating patterns with controlled color symmetry can inspire innovative fabric motifs and decorative materials. In education, visually rich hyperbolic colorings serve as powerful tools for teaching abstract concepts in geometry, group theory, and symmetry, making advanced mathematics more accessible through visualization. By bridging algebraic theory, geometry, and real-world applications, this research contributes to both mathematical understanding and interdisciplinary innovation.

Methods

This study employed a structural investigation on colorings induced by low-index subgroups of hyperbolic triangle group *732. The methodology involved two main stages: group generation and hyperbolic coloring construction. First, the hyperbolic triangle group *732 was defined and implemented using the software GAP (Groups, Algorithms, and Programming). GAP allowed the group to be represented algebraically with its standard generators and relations. Using GAP, the low-index subgroups of *732 were systematically generated, providing the algebraic structures necessary to produce coset partitions. Next, the study constructed right coset colorings corresponding to these low-index subgroups. The method involved applying the 3n method to assign colors to tiles in the hyperbolic plane based on the subgroup generators. Each right coset of a subgroup was associated with a unique color, so that all tiles corresponding to elements in the same coset shared the same color. This produced a subgroup-induced coloring of the hyperbolic tiling, ensuring that the coloring respected the symmetry properties of the subgroup. Finally, the properties of the low-index subgroups, such as index, normality, and generator structure, were analyzed in relation to their induced colorings. This approach allowed the study to link the algebraic properties of subgroups with the visual and geometric patterns in hyperbolic tilings, demonstrating how group-theoretic concepts can be translated into tangible color structures in the hyperbolic plane.

Results and Discussion

Subgroup Structure of *732

The H^2 or the hyperbolic plane is a unique and fascinating model of non-Euclidean geometry where the parallel postulate of Euclidean geometry does not hold. Unlike the familiar flat surfaces, the hyperbolic plane has a constant negative curvature, creating a surface that appears saddle-shaped. This curvature gives rise to intriguing properties and behaviors, such as the fact that the sum of angles in a hyperbolic triangle is always less than 180^0, and parallel lines can diverge away from each other. Various models, like the Poincaré disk and the upper half-plane model, help visualize and study the properties of the hyperbolic plane, making it an essential concept in advanced mathematics, particularly in the fields of geometry, topology, and theoretical physics. Understanding the hyperbolic plane provides deep insights into the structure of space, offering a broader perspective beyond the conventional Euclidean framework. A computer program, GAP (Group, Algorithms and Programming), was used to list generators of the subgroups of the group *732 with an index less than or equal to 24. The triangles that represent the elements of the group's subgroups are colored using 3n Precise Coloring as shown in Figures 1 to 9. The components of the low-index subgroups of the hyperbolic triangle group *732 were identified using right coset coloring. Using Conway notations, the subgroups were categorized based on their symmetry structure.

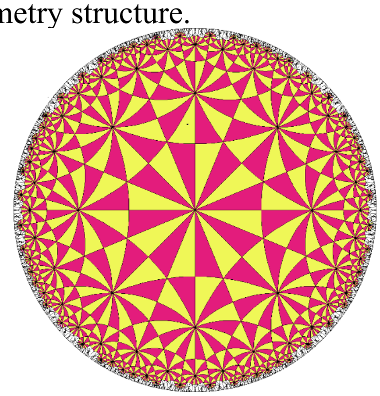

Figure 1. Coloring using right cosets of the subgroups of *732 of Index 2.

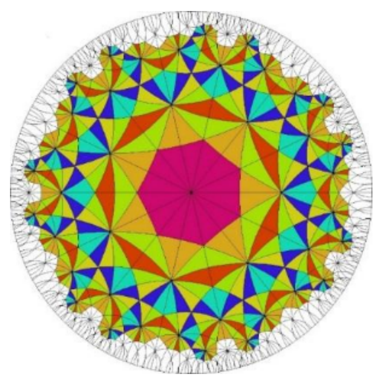

Figure 2. Coloring using right cosets of the subgroups of *732 of Index 8.

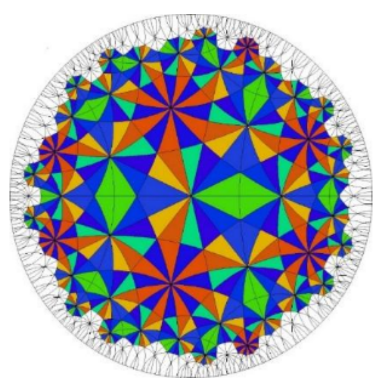

Figure 3. Coloring using right cosets of the subgroups of *732 of Index 9.

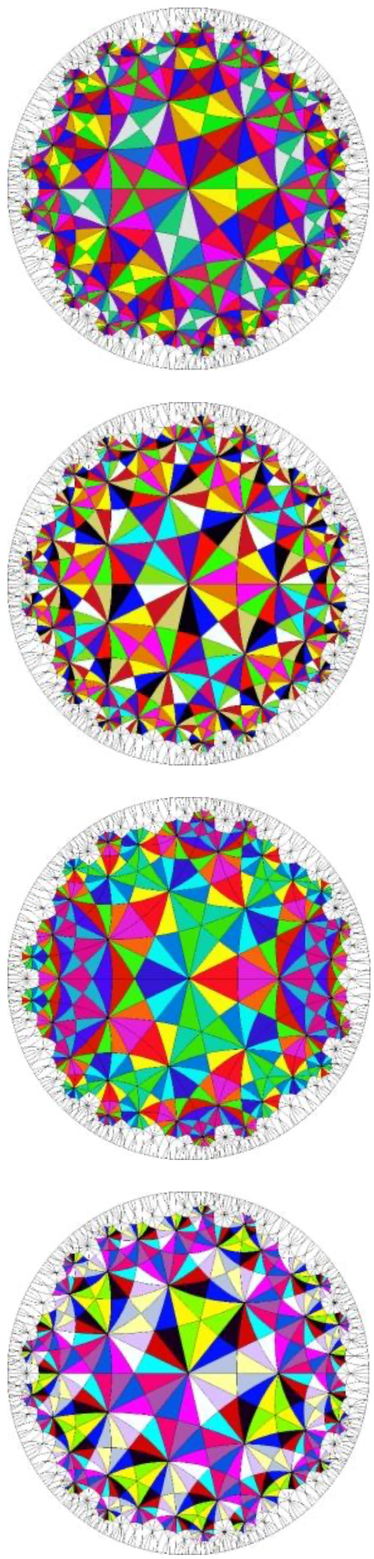

Figure 4. Coloring using right cosets of the subgroups of *732 of Index 14.

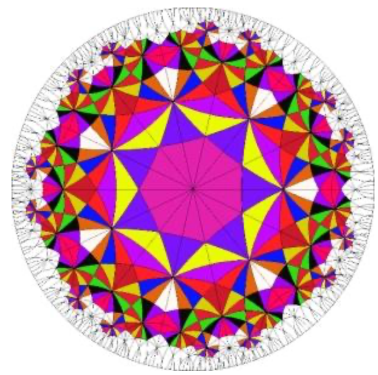

Figure 5. Coloring using right cosets of the subgroups of *732 of Index 15.

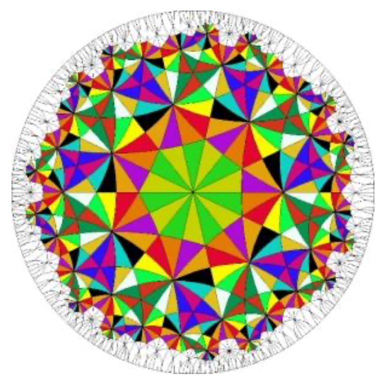

Figure 6. Coloring using right cosets of the subgroups of *732 of Index 16.

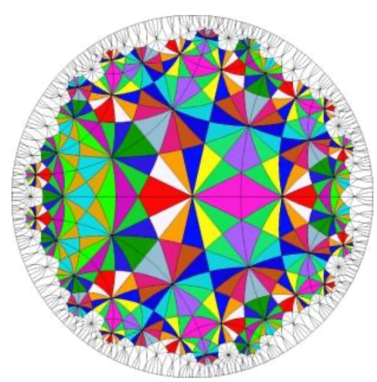

Figure 7. Coloring using right cosets of the subgroups of *732 of Index 21.

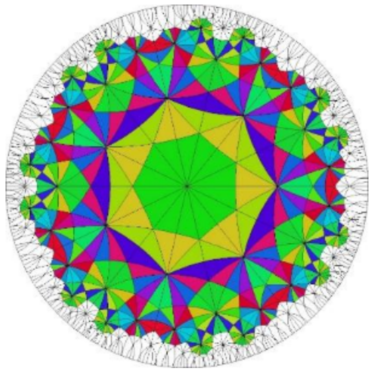

Figure 8. Coloring using right cosets of the subgroups of *732 of Index 22

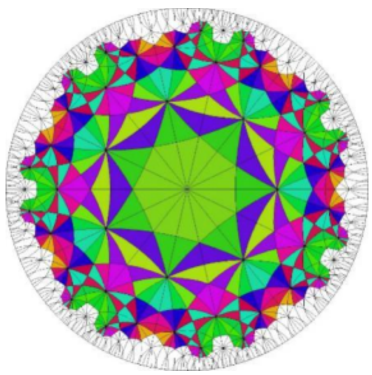

Figure 9. Coloring using right cosets of the subgroups of *732 of Index 24

In investigating the symmetry structures of the low-index subgroups of the hyperbolic triangle group *732, the fundamental region of each subgroup was first identified. For each induced coloring, the fundamental region is defined as a closed and bounded area consisting of one triangle from each color class in the tiling. This region represents the smallest unit from which the entire colored hyperbolic tiling can be generated through the action of the subgroup. Identifying the fundamental region is essential, as it reveals both the geometric and algebraic structure underlying the coloring. Figures 1–9 illustrate examples of right coset colorings obtained from selected low-index subgroups of *732. In each figure, the application of the 3n coloring method assigns colors to the triangles corresponding to subgroup elements, ensuring that triangles belonging to the same right coset share the same color. These figures demonstrate how the subgroup structure determines the repetition and arrangement of colors across the hyperbolic plane. In particular, the symmetry observed in each figure reflects the generators and relations of the corresponding subgroup. Right coset coloring proved effective in identifying and distinguishing low-index subgroups of *732. By examining the resulting color distributions, the subgroups were classified according to their symmetry structures using Conway notation. This classification highlights how reflectional and rotational symmetries are preserved or broken depending on subgroup properties. Similar observations were reported in earlier studies on hyperbolic colorings, where subgroup-induced partitions were shown to encode symmetry information in a visually meaningful way (Hernandez, 2003; Rigby, 1997).

The findings of this study are consistent with previous work on precise and perfect colorings of hyperbolic tilings. For instance, Hernandez and Felix (2008) demonstrated that subgroup actions on {3,n}{3,n} hyperbolic tilings lead to well-defined color symmetries, while Yao and Hernandez (2012) showed that the 3n method produces systematic and symmetric colorings. The present study extends these results by explicitly relating low-index subgroup structures of *732 to Conway symmetry notation, thereby strengthening the link between algebraic classification and geometric visualization. Understanding subgroup structures and their induced coset colorings provides a framework for generating complex yet structured patterns applicable to textile design, architectural ornamentation, and digital visualization. Moreover, the clear correspondence between algebraic properties and visual outcomes supports the use of hyperbolic colorings as educational tools for teaching abstract concepts in group theory and non-Euclidean geometry. These results also open pathways for future research on other hyperbolic triangle groups and the development of computational tools for automated pattern generation.

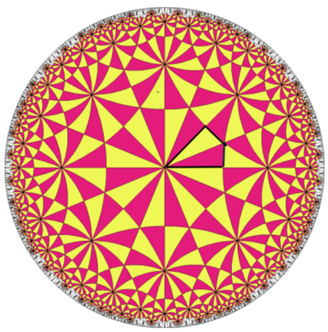

Figure 10. Fundamental region for the right coset coloring of a generator of RQ, PQ subgroup of *732 of Index 2.

In the context of hyperbolic geometry, Figure 10 illustrates the fundamental region for the right coset coloring of a subgroup generated by RQPQ within the reflection group ∗732 with an index of 2. The group ∗732 corresponds to a hyperbolic triangle with angles π/7, π/3, and π/2, which tiles the hyperbolic plane through repeated reflections. Considering the subgroup generated by RQ,PQ, the hyperbolic plane is partitioned into two distinct cosets, resulting in a fundamental region composed of two hyperbolic triangles. The right coset coloring process assigns a different color to each triangle according to its coset membership, creating a visual representation that clearly reflects the subgroup's structure within ∗732. Notably, although the subgroup divides the plane into separate cosets, the overall symmetry and tiling patterns are still governed by ∗732. This demonstrates that the geometric properties of the original reflection group continue to shape the configuration, highlighting the interplay between subgroup algebraic structure and hyperbolic plane symmetries.

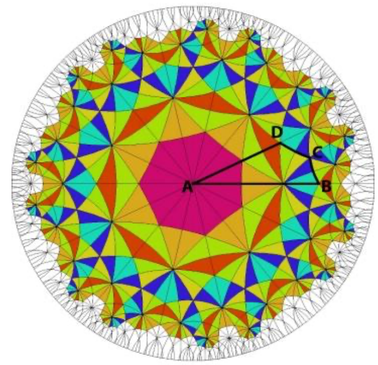

Figure 11. Fundamental region for the right coset coloring of a generator of Q, R, PRQRQPRQRQRP subgroup of *732 of Index 8.

As shown in Figure 11, the subgroup of ∗732∗732 generated by the elements Q,R,PRQRQPRQRQRP provides a detailed view of the symmetrical properties of its fundamental region. When subjected to right coset coloring, this fundamental area produces a complex tiling of the hyperbolic plane. Notably, the region exhibits two distinct rotational symmetries: a three-fold rotation around points labeled C, allowing rotation by multiples of 2π/3, and a seven-fold rotation around points labeled A, permitting rotation by multiples of π/7. In contrast, points labeled B and D display only trivial one-fold symmetry, remaining invariant under these rotations. This analysis of rotational symmetries is succinctly captured by the Conway notation 3**7, which encodes the subgroup's rotational characteristics and provides a compact summary of its symmetry structure within the hyperbolic plane. The visual representation in Figure 11 thus illustrates how subgroup-generated tilings reflect both the complexity and the inherent symmetry of $*732$, offering insights relevant to theoretical exploration.

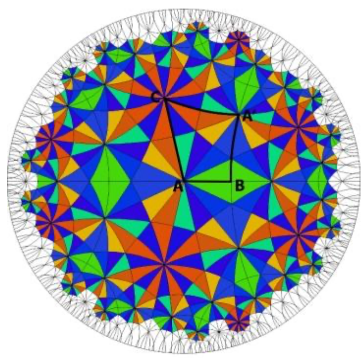

Figure 12. A fundamental region for the right coset coloring of a generator Q, P, RQPRQRQRQ of subgroup of *732 of Index 9

The subgroup of $*732$ generated by the elements Q, P, and RQPRQRQRQ exhibits intricate symmetrical patterns in its fundamental region, as illustrated in Figure 12. The hyperbolic plane is tessellated in this region using right coset coloring, revealing the subgroup's structure through distinct rotational symmetries. Notably, there is a seven-fold rotation around points labeled C, allowing rotations by multiples of $\pi/2$; a two-fold rotation around points BB, permitting rotations of π; and points labeled AA′, which display trivial one-fold symmetry, remaining invariant under transformations. These rotational characteristics are succinctly captured by the Conway notation **72, which encodes the interplay of the subgroup's symmetries in a compact form. The visualization demonstrates how low-index subgroups of $*732$ can generate complex yet structured tilings.

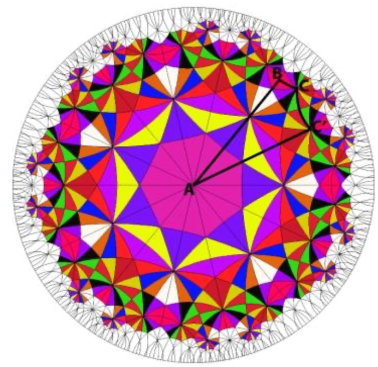

Figure 13. Fundamental region for the right coset coloring of a generator of Q, R, PRQRQPRQRPRQRQRP subgroup of *732 Index 15.

For the subgroup of $*732$ generated by Q,R,PRQRQPRQRPRQRQRP with an index of 15, Figure 13 illustrates the fundamental region used in the right coset coloring. Understanding the complex symmetries of hyperbolic tilings requires careful analysis of the rotational and reflectional symmetries present in this basic area, which the subgroup analysis highlights. This subgroup combines rotations and reflections, concisely represented by the Conway notation $*7{*}{*}2$. The $*7$ component indicates a seven-fold rotational symmetry, while the $**2$ component captures additional symmetries such as reflections or glide reflections, creating a more intricate structure. By examining the right cosets and their associated colorings, one can visualize how these symmetries propagate to tessellate the hyperbolic plane, producing a rich and repeating pattern. This analysis not only deepens understanding of the algebraic and geometric properties of hyperbolic groups but also illustrates the complexity of non-Euclidean symmetry, with potential applications in pattern generation.

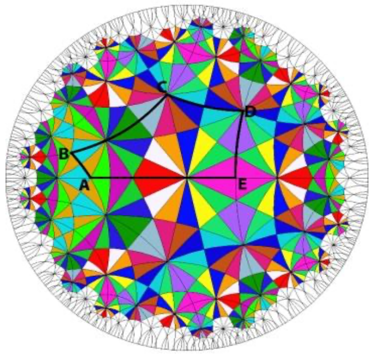

Figure 14. A fundamental region for the right coset coloring of a generator Q, P, RQPRQRPRQRRQRQRQPRQRPQRQRP RQRRQR subgroup of *732 Index 21.

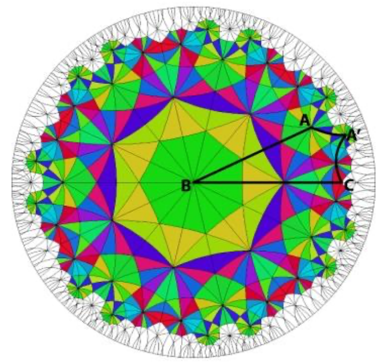

Figure 15. Fundamental region for the right coset coloring of a generator of Q, R, PRQRQPRQRPQRQRPQRQRP subgroup of *732 of Index 22.

Figure 14 illustrates the fundamental region for the right coset coloring of the subgroup generated by Q, P, RQPRQRPRQR,RQRQRQPRQRPQRQRPR QRRQR, within the ∗732 symmetry group, which has an index of 21. This subgroup exhibits a highly intricate combination of reflections and rotations, resulting in complex symmetry within the hyperbolic plane. Its symmetry is succinctly represented using Conway notation ∗∗∗∗2, where the multiple asterisks indicate layered reflection symmetries and the final 22 denotes a two-fold rotational symmetry. Consequently, the fundamental region is divided into 21 distinct parts, each transformed according to the subgroup's elements. These transformations, including reflections (Q), rotations (P), and their combinations, produce a richly patterned and highly symmetric tiling. Applying right coset coloring allows each coset to be assigned a distinct color, making the subgroup's actions visible and highlighting how the fundamental region maps onto itself under the subgroup transformations. This visualization demonstrates how the subgroup's structure ensures that the pattern repeats consistently and symmetrically across the hyperbolic plane. Overall, the analysis underscores the depth of hyperbolic geometry, revealing complex symmetrical patterns that emerge naturally from simple geometric operations.

Within the ∗732 symmetry group, which has an index of 22, Figure 15 illustrates the fundamental region for the right coset coloring of the subgroup generated by Q,R,PRQRQPRQRPQRQRPQRQR P. This subgroup comprises a sequence of transformations, including rotations, reflections, and their combinations, producing a complex and highly structured pattern on the hyperbolic plane. The subgroup's symmetries are succinctly summarized using Conway notation ∗∗∗7, where the three asterisks indicate multiple layers of reflection symmetries, and the 7 denotes a seven-fold rotational symmetry. As a result, the fundamental region is divided into 22 distinct parts, each transformed according to the subgroup elements, yielding a highly repetitive and symmetric tiling. Applying right coset coloring allows each coset to be assigned a distinct color, making it visually clear how the subgroup's transformations map different sections of the fundamental region onto one another. This approach not only highlights the preservation and propagation of symmetry across the hyperbolic plane but also provides an intuitive means of understanding the intricate patterns by LI subgroups.

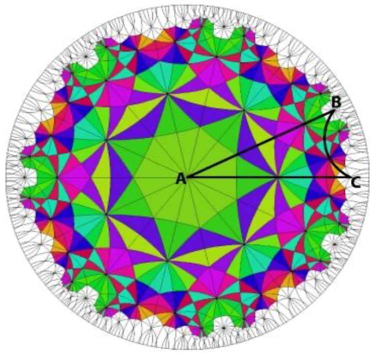

Figure 16. Fundamental region for the right coset coloring of a generator of Q, R, PRQRQRPRQRQRPRQRQRP subgroup of *732 of Index 24.

Figure 16 illustrates the fundamental region for the right coset coloring of the subgroup generated by Q,R,PRQRQRPRQRQRPRQRQRP within the ∗732 symmetry group, which has an index of 24. This subgroup consists of a combination of reflections and rotations, which are succinctly represented by Conway notation ∗∗7. The two asterisks indicate two layers of reflection symmetries, while the 7 denotes a seven-fold rotational symmetry, resulting in a complex tiling pattern on the hyperbolic plane. Applying right coset coloring assigns distinct colors to each coset, visually highlighting how the subgroup elements map different parts of the fundamental region onto one another. This method ensures that the resulting pattern is both consistent and symmetric, repeating systematically across the hyperbolic plane.

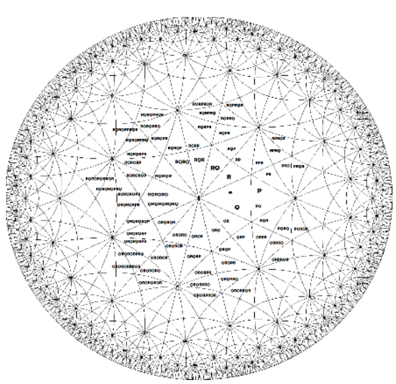

Figure 17. Triangle group of *732

In the hyperbolic plane, as illustrated in Figure 17, the process of reflecting or inverting a given triangle across its sides generates a sequence of triangles, each derived iteratively from the previous one. Within this framework, the triangle QRP emerges as a key component of the geometric construction. Unlike Euclidean triangles, the angles of QRP follow hyperbolic trigonometric relationships, resulting in a sum of angles. Despite this departure from Euclidean norms, QRP plays a crucial role in understanding the topology and tiling patterns of the hyperbolic plane. Its placement within the iterative reflection process ensures that the hyperbolic plane is fully covered without gaps or overlaps. Moreover, the triangle inherits symmetry both from the original generating triangle and from the iterative transformations, illustrating the interplay between symmetry and complexity in hyperbolic geometry. Studying triangle QRP thus provides insights into the fundamental principles of hyperbolic tiling, revealing how non-Euclidean transformations produce rich, repeating patterns and offering a deeper understanding of geometric structures that differ significantly from those of classical Euclidean spaces.

Conclusion

The significant structural information can be obtained from colorings induced by right cosets of subgroups. While this study focused on the hyperbolic triangle group *732, future research may extend the methodology to other hyperbolic triangle groups to compare subgroup structures and induced coloring behaviors across different group parameters. Such comparative analyses may contribute to a broader classification of hyperbolic coset colorings. Further studies may also apply right coset coloring techniques to Euclidean and spherical symmetry groups, enabling direct comparisons among geometries of different curvature and providing deeper insight into the relationship between geometry, symmetry, and coloring. In addition, alternative methods for constructing hyperbolic tilings may be explored. Beyond Conway's symmetry notation and Rigby's k-tree constructions, this study highlights the feasibility of generating tilings by joining the vertices of a subgroup's fundamental region through a single complete rotation. Investigating this approach further may lead to simpler and more efficient tiling constructions. Finally, future research may employ advanced computational tools by integrating GAP with visualization software or developing dedicated applications for generating and analyzing induced colorings. Such tools could enhance pattern visualization, improve computational efficiency, and broaden applications in design, architecture, and mathematics education.

Acknowledgments

The author sincerely thanks Dr. Lawrence A. Eclarin for her guidance and support throughout this research. She also acknowledges the valuable comments of the research panel, Prof. Raymond Paul L. Eclarin, Dr. Michelle D. Reynera, and Prof. Emely P. Tanagon. Appreciation is also extended to the Mariano Marcos State University for the support.

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